LECTURE NOTES: 4-4 INDETERMINATE FORMS AND L'HOSPITAL'S RULE (PART 1)

MOTIVATING EXAMPLES: Evaluate the Chapter 2 limits below, justifying each step:

a) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-5x+6} = \lim\limits_{x\to 2} \dfrac{(x+2)(x-2)}{(x-3)(x-2)}$ algebra

$= \lim\limits_{x\to 2} \dfrac{x+2}{x-3}$ for $x\neq 2$, these are equal

$= \dfrac{4}{-1} = -4$ plug in

b) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ b/c we can prove this using a clever geometric argument.

L'Hospital's Rule If a limit has the form $\frac{0}{0}$ or $\pm\infty/\pm\infty$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right actually exists.

L'Hospital's allows $x \to \pm\infty$

QUESTION 1: Determine whether or not l'Hospital's Rule applies to the MOTIVATING EXAMPLES (copied below) and if it does, apply it. Do you get the same answer?

the "H" indicates when we used l'Hospital's Rule

a) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-5x+6}$ ← has the form $\frac{0}{0}$.

$\overset{H}{=} \lim\limits_{x\to 2} \dfrac{2x}{2x-5} = \dfrac{4}{-1} = -4$

b) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ ← has the form $\frac{0}{0}$

that "H" again!

$\overset{H}{=} \lim\limits_{x\to 0} \dfrac{\cos x}{1} = \dfrac{1}{1} = 1$

same answer as before

QUESTION 2: Why does l'Hospital's Rule work?

If $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ has form $\frac{0}{0}$, then $f(a)=0$ and $g(a)=0$.

So $\dfrac{f(x)}{g(x)} = \dfrac{f(x)-0}{g(x)-0} = \dfrac{f(x)-f(a)}{g(x)-g(a)} = \dfrac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$. Now, look at the limits of each end of this string of equalities :)

provided $x\neq a$

PRACTICE PROBLEMS: Evaluate the following limits.

1. $\lim_{x\to 0} \frac{\tan(5x)}{\sin(3x)}$ form $\frac{0}{0}$

$\stackrel{H}{=} \lim_{x\to 0} \frac{5\sec^2(5x)}{3\cos(3x)}$

$= \frac{5}{3}$

2. $\lim_{u\to\infty} \frac{e^{u/10}}{u^2}$ form $\frac{\infty}{\infty}$

$\stackrel{H}{=} \lim_{u\to\infty} \frac{\frac{1}{10}e^{u/10}}{2u}$ still in form $\frac{\infty}{\infty}$

$\stackrel{H}{=} \lim_{u\to\infty} \frac{\frac{1}{100}e^{u/100}}{2} = \infty$

because $e^{u/100} \to \infty$ as $u \to \infty$

Lesson: Sometimes you have to apply l'Hospital's Rule more than once.

3. $\lim_{x\to 0} \frac{\cos(4x)}{e^{2x}} = \frac{\cos(0)}{e^0} = \frac{1}{1} = \underline{1}$

L'Hospital's Rule doesn't apply.

Lesson: Always check that L'Hospital's Rule applies before using it.

4. $\lim_{x\to 0} \frac{xe^x}{2^x - 1}$ form $\frac{0}{0}$

$\stackrel{H}{=} \lim_{x\to 0} \frac{e^x + xe^x}{(\ln 2)\, 2^x} = \frac{1}{\ln 2}$

5. $\lim_{x\to 1^+} \left(\ln(x^4-1) - \ln(x^9-1)\right)$ form $\infty - \infty$

$$= \lim_{x\to 1^+} \ln\left(\frac{x^4-1}{x^9-1}\right)$$ algebra

$$= \ln\left[\underbrace{\lim_{x\to 1^+}\left(\frac{x^4-1}{x^9-1}\right)}_{\text{form } \frac{0}{0}}\right]$$ optimism + rules about limits

$$\overset{H}{=} \ln\left[\lim_{x\to 1^+} \frac{4x^3}{9x^8}\right]$$

$$= \ln\left[4/9\right]$$ ✓

6. $\lim_{x\to\infty} \sqrt{x}e^{-x/2}$ ← form $\infty \cdot 0$ (but we easily turn the "0" term into an "∞" term w/ algebra.)

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}}$$ ← Now it has the form $\frac{\infty}{\infty}$.

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{\frac{1}{2}e^{x/2}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}\,e^{x/2}} = 0$$ b/c $\sqrt{x}\,e^{x/2} \to \infty$ as $x \to \infty$

★ Articulate explicitly what *trick* was used to evaluate the last limit and state precisely what sort of limits this trick will apply to in general.

If a limit has the form $0 \cdot \infty$, one of the terms in the product can be written in the denominator so that the expression has the form

$\frac{0}{0}$ or $\frac{\infty}{\infty}$ ← rewrite "0" term

rewrite "∞" term